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Galilean invariance and stabilized methods for compressible flows

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SUMMARY

In a recent work (*Comput. Methods Appl. Mech. Eng.* 2007; **196**(4–6):966–978), it was observed that lack of Galilean invariance led to catastrophic instabilities when stabilized methods were used in Lagrangian shock hydrodynamics computations. By means of an arbitrary Lagrangian–Eulerian (ALE) formulation, Galilean invariant SUPG operators were consistently derived in (*Comput. Methods Appl. Mech. Eng.* 2007; **196**(4–6):1108–1132), and their Lagrangian and Eulerian limits were compared to the most commonly used stabilized formulations. In the particular case of Eulerian meshes, it was shown that most of the SUPG operators designed to date for compressible flow computations *are not* invariant. However, due to the significant overhead of algebraic manipulations, the use in (*Comput. Methods Appl. Mech. Eng.* 2007; **196**(4–6):1108–1132) of the referential form of the ALE equations made the presentation of the main ideas quite involved. The present paper addresses this particular issue, since the invariance analysis is presented with the aid of the intuitive current configuration reference frame, more familiar to computational fluid dynamicists. Copyright © 2007 John Wiley & Sons, Ltd.

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1. INTRODUCTION

The concept of Galilean invariance is a particularization of the more general *objectivity principle*, which states that certain properties of mechanical systems must remain invariant under coordinate transformations. In a Galilean transformation, a constant velocity *shift* is applied to the origin of the spatial coordinates. In general, the Galilean invariance principle is satisfied

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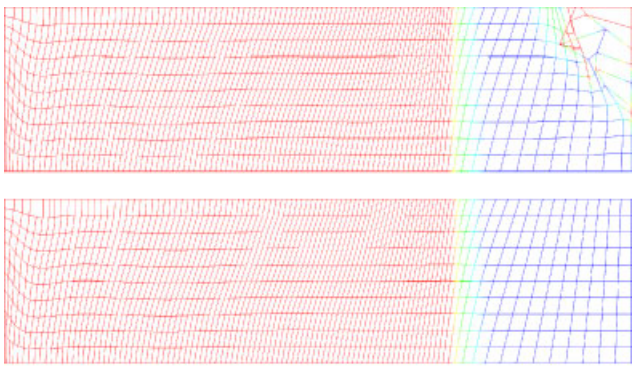

Figure 1. Results from the computations in [16]. Mesh distortion plot: The colour scheme represents the pressure. Above: SUPG formulation violating Galilean invariance. Below: SUPG abiding the Galilean invariance principle. A classical quadrilateral Saltzmann mesh is used in an implosion computation. The initial velocity is of unit magnitude and directed horizontally from right to left, except the left boundary which is held fixed. The initial density is unity and the initial specific internal energy is 10^{-1}. A shock forms at the left boundary and advances to the right. Note the *mesh coasting* phenomenon on the top right corner of the upper domain, absent in the SUPG formulation satisfying Galilean invariance, below.

by the equations governing the motion of *continua*. In numerical computations, it is advisable that the discrete counterpart of the continuum equations maintain such property. This is typically the case for Bubnov– and Petrov–Galerkin finite element methods, which enforce *orthogonality* between the equations of motion (namely, the *Galerkin residual*) and the test function space. In this case, the Galilean principle translates into the requirement that—under Galilean transformations—the residual remains orthogonal to the Bubnov– and Petrov–Galerkin test spaces. Because the shift by a constant velocity factors out of all the integrals in the variational statement, it is straightforward to prove that if the equations of the continuum are invariant, so are the discrete equations generated by a Bubnov– or Petrov–Galerkin method (see Section 5).

SUPG and variational multiscale stabilized methods [1–8] are Petrov–Galerkin methods in which the *local* structure of the partial differential equations is used to perturb the Bubnov–Galerkin test space. In this sense, stabilized methods are a more general class of Petrov–Galerkin methods. Classical Petrov–Galerkin methods require the test space to be chosen *a priori*: The test function basis does not change with the parameters in the equations to be simulated, such as *local* Péclet number, *local* Reynolds number, etc. SUPG methods, instead, are *locally/physically* adapted Petrov–Galerkin methods, in which a local, parameter-dependent perturbation of the test function space is introduced to improve the overall stability properties of the underlying Bubnov–Galerkin formulation.

Hence, invariance of the perturbed test space is crucial to avoid the paradox of having the stability properties of the method depending on the *observer*. As was shown in [9], while Galilean invariance is respected by virtually all stabilized methods for incompressible flows [1, 10–15], this is not the case for the *large majority* of stabilized compressible flow computations on Eulerian (fixed) meshes. As a result, instabilities were generated (see, e.g. Figure 1) when commonly used stabilization procedures were applied to compressible Lagrangian hydrodynamics computations [16].

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The present article complements the work in [9] aimed to analyse and obviate this problem, by presenting the discussion in the current configuration rather than the referential configuration. The current configuration frame is in fact more intuitive, and prevents algebraic details to obscure the key points of the analysis. With respect to standard approaches, also the present formulation delivers a conspicuous reduction in the computational cost of the stabilization operator.

The rest of the material is organized as follows: A very general discussion of the issue of Galilean invariance in the context of arbitrary Lagrangian–Eulerian (ALE) equations and its Eulerian and Lagrangian limits is presented in Section 2. The ALE description of the kinematics of motion is developed in Section 3. A stabilized space–time variational formulation of the ALE compressible Euler equations is developed in Section 4. Section 5 presents an analysis of the invariance properties of the residuals and their effect on the approximation to the subgrid-scale solution. In Section 6, a Galilean consistency analysis shows that standard SUPG formulations for compressible flows yield a non-invariant test function space. A new, invariant approach is also developed, and its advantages are analysed in detail. Conclusions are summarized in Section 7.

2. GALILEAN TRANSFORMATIONS

A Galilean transformation can be expressed by the *affine* mapping

$$G : \mathbb{R}^+ \times \mathbb{R}^{n_d} \times \mathbb{R}^{n_d} \longrightarrow \mathbb{R}^+ \times \mathbb{R}^{n_d} \times \mathbb{R}^{n_d} \tag{1}$$

$$[t \ \mathbf{x}^{\mathrm{T}} \ \mathbf{v}^{\mathrm{T}}]^{\mathrm{T}} \mapsto [\tilde{t} \ \tilde{\mathbf{x}}^{\mathrm{T}} \ \tilde{\mathbf{v}}^{\mathrm{T}}]^{\mathrm{T}} \tag{2}$$

and consists of a *shift* in the spatial coordinate by $\mathbf{V}^{\mathrm{G}}t$, namely,

$$\tilde{t} = t \tag{3}$$

$$\tilde{\mathbf{x}} = \mathbf{x} - \mathbf{V}^{\mathrm{G}}t \tag{4}$$

$$\tilde{\mathbf{v}} = \mathbf{v} - \mathbf{V}^{\mathrm{G}} \tag{5}$$

Here n_d represents the number of space dimensions. Galilean transformations are commonly used to verify the consistency of physical and computational models. A well-designed, consistent model must be Galilean invariant, or, more precisely, its functional form $\mathscr{M}$ has to transform as

$$\mathscr{M}(\mathbf{v}, \mathbf{x}, t, \ldots) \xrightarrow{\mathrm{G}} \mathscr{M}(\tilde{\mathbf{v}}, \tilde{\mathbf{x}}, \tilde{t}, \ldots) \tag{6}$$

A finite element method is generally developed over a geometrical model, by means of the computational grid or *mesh*, a discrete subdivision of the physical space. The mesh may be fixed (Eulerian), may follow the material motion (Lagrangian), or may have an arbitrary motion (ALE). In the last case, it is straightforward to observe that the field $\mathbf{c}$, the difference of the material velocity $\mathbf{v}$ and the mesh velocity $\hat{\mathbf{v}}$, is invariant under Galilean transformations (see Figure 2). This simple observation clearly implies that, for an ALE formulation, an invariant SUPG perturbation to the Bubnov–Galerkin test function *can only* depend on thermodynamic variables and their gradients, the velocity $\mathbf{c}$, derivatives (in space or time) of the material velocity, and gradients of the position vector.

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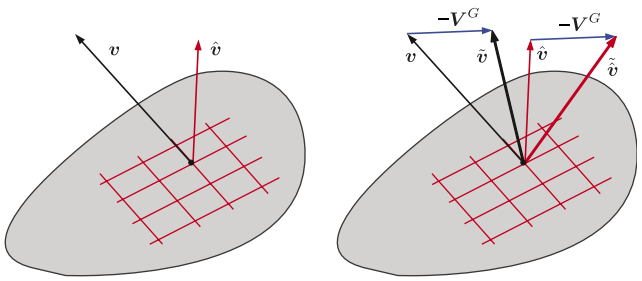

Figure 2. Sketch of a Galilean transformation for a generic ALE mesh. Left: A material domain, and the corresponding mesh, are moving with velocity $\mathbf{v}$ and $\hat{\mathbf{v}}$, respectively. Left: After a Galilean transformation is applied, the material and the mesh are moving with velocities $\tilde{\mathbf{v}} = \mathbf{v} - \mathbf{V}^{\mathrm{G}}$ and $\tilde{\hat{\mathbf{v}}} = \hat{\mathbf{v}} - \mathbf{V}^{\mathrm{G}}$, respectively. The relative velocity of the material with respect to the mesh is an invariant: $\tilde{\mathbf{c}} = \tilde{\mathbf{v}} - \tilde{\hat{\mathbf{v}}} = \mathbf{v} - \mathbf{V}^{\mathrm{G}} - \hat{\mathbf{v}} + \mathbf{V}^{\mathrm{G}} = \mathbf{v} - \hat{\mathbf{v}} = \mathbf{c}$.

Remarks

(i) For a Lagrangian mesh, $\mathbf{c} = \mathbf{0}$, and the SUPG term can only be a function of the thermodynamic state, the material properties of the system, velocity derivatives (again, in space or time), and gradients of the position vector.
(ii) After a Galilean change of coordinates is performed, an Eulerian mesh transforms into a mesh moving with uniform velocity $-\mathbf{V}^{\mathrm{G}}$.
(iii) Developing SUPG operators for Eulerian meshes is somewhat problematic, since it is not possible to discern from the equations whether the meaning of '$\mathbf{v}$' is $\mathbf{v} - \hat{\mathbf{v}} = \mathbf{v} - \mathbf{0} = \mathbf{c}$, a relative velocity, or simply $\mathbf{v}$, the absolute material velocity. A more consistent approach is to start from the ALE formulation and then take the limit for a fixed (Eulerian) mesh.

3. ARBITRARY LAGRANGIAN–EULERIAN KINEMATICS

In what follows, the notation used in [17–19] is adopted, with minor differences. A point of departure in the discussion of the arbitrary Lagrangian–Eulerian approach is to define the *material* (or *Lagrangian*), *referential*, and *Eulerian* reference frames. Let Ω_0, $\hat{\Omega}$, and Ω be open sets in $\mathbb{R}^{n_d}$ (see Figure 3). The *deformation* $\boldsymbol{\varphi}$ is the transformation from the material to the Eulerian reference frame

$$\boldsymbol{\varphi} : \Omega_0 \to \Omega = \boldsymbol{\varphi}(\Omega_0) \tag{7}$$

$$\mathbf{X} \mapsto \mathbf{x} = \boldsymbol{\varphi}(\mathbf{X}, t) \quad \forall \mathbf{X} \in \Omega_0, \quad t \geqslant 0 \tag{8}$$

Here, $\mathbf{X}$ is the material coordinate (which usually corresponds to the point vector in the initial configuration of the body) and $\mathbf{x}$ is the point vector in the Eulerian frame. Ω_0 is the domain occupied by the body in the material reference frame. $\boldsymbol{\varphi}$ maps Ω_0 to Ω, the domain occupied by the body in the current configuration (Eulerian frame). It is also useful to define the *deformation*

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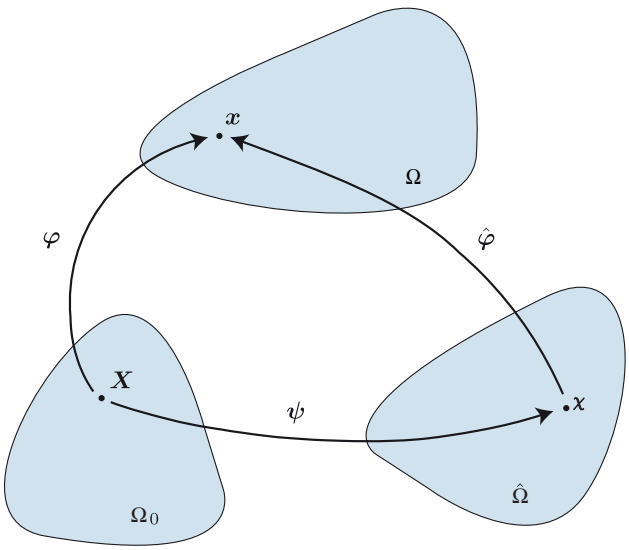

Figure 3. Sketch of the maps $\boldsymbol{\varphi}$, $\hat{\boldsymbol{\varphi}}$, and $\boldsymbol{\psi}$ for the generalized ALE framework.

gradient and the *Jacobian determinant*:

$$\mathbf{F} = \nabla_{\mathbf{x}}\boldsymbol{\varphi} = \frac{\partial \varphi_i}{\partial X_j} = \frac{\partial x_i}{\partial X_j} \tag{9}$$

$$J = \det(\mathbf{F}) \tag{10}$$

The *referential map* $\hat{\boldsymbol{\varphi}}$, from the referential frame to the Eulerian frame, is defined as

$$\hat{\boldsymbol{\varphi}} : \hat{\Omega} \to \Omega = \hat{\boldsymbol{\varphi}}(\hat{\Omega}) \tag{11}$$

$$\boldsymbol{\chi} \mapsto \mathbf{x} = \hat{\boldsymbol{\varphi}}(\boldsymbol{\chi}, t) \quad \forall \boldsymbol{\chi} \in \hat{\Omega}, \quad t \geqslant 0 \tag{12}$$

where $\boldsymbol{\chi}$ is the point vector in the referential frame. $\hat{\Omega}$, the domain occupied by the body in the referential frame, is mapped to Ω by $\hat{\boldsymbol{\varphi}}$. In addition, the *mesh deformation gradient* and the *mesh Jacobian determinant* are defined as

$$\hat{\mathbf{F}} = \nabla_{\boldsymbol{\chi}}\hat{\boldsymbol{\varphi}} = \frac{\partial \hat{\varphi}_i}{\partial \chi_j} = \frac{\partial x_i}{\partial \chi_j} \tag{13}$$

$$\hat{J} = \det(\hat{\mathbf{F}}) \tag{14}$$

The referential frame of reference lies on a mesh which is not fixed in space (Eulerian) nor attached to the material (Lagrangian), but moves in time with an arbitrary motion. The transformation from the material to the referential frame will also be needed, namely

$$\boldsymbol{\psi} : \Omega_0 \to \hat{\Omega} = \boldsymbol{\psi}(\Omega_0) \tag{15}$$

$$\mathbf{X} \mapsto \boldsymbol{\chi} = \boldsymbol{\psi}(\mathbf{X}, t) \quad \forall \mathbf{X} \in \Omega_0, \quad t \geqslant 0 \tag{16}$$

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The definition of the *referential deformation gradient* reads

$$\nabla_{\mathbf{x}}\boldsymbol{\psi} = \frac{\partial \psi_i}{\partial X_j} = \frac{\partial \chi_i}{\partial X_j} \tag{17}$$

Displacements are defined as

$$\mathbf{u} = \boldsymbol{\varphi}(\mathbf{X}, t) - \boldsymbol{\varphi}(\mathbf{X}, 0) = \mathbf{x}(\mathbf{X}, t) - \mathbf{X} \tag{18}$$

$$\hat{\mathbf{u}} = \hat{\boldsymbol{\varphi}}(\boldsymbol{\chi}, t) - \hat{\boldsymbol{\varphi}}(\boldsymbol{\chi}, 0) = \mathbf{x}(\boldsymbol{\chi}, t) - \boldsymbol{\chi} \tag{19}$$

with the practical assumption, $\boldsymbol{\chi}(\mathbf{X}, t = 0) = \mathbf{X}$. The referential displacement $\hat{\mathbf{u}}$ is the displacement undergone by the mesh. Analogously, material and mesh velocities can be defined:

$$\mathbf{v} = \left.\frac{\partial \boldsymbol{\varphi}}{\partial t}\right|_{\mathbf{X}} = \left.\frac{\partial \mathbf{u}}{\partial t}\right|_{\mathbf{X}} = \dot{\mathbf{u}} \tag{20}$$

$$\hat{\mathbf{v}} = \left.\frac{\partial \hat{\boldsymbol{\varphi}}}{\partial t}\right|_{\boldsymbol{\chi}} = \left.\frac{\partial \mathbf{u}}{\partial t}\right|_{\boldsymbol{\chi}} \tag{21}$$

Using the chain rule, it is possible to derive an important expression for the Lagrangian time derivative of a scalar-valued function f:

$$\begin{aligned} \dot{f}(\boldsymbol{\chi}, t) &= \left.\frac{\partial f}{\partial t}\right|_{\boldsymbol{\chi}} + \nabla_{\boldsymbol{\chi}} f \cdot \left.\frac{\partial \boldsymbol{\psi}(\mathbf{X}, t)}{\partial t}\right|_{\mathbf{X}} \\ &= \left.\frac{\partial f}{\partial t}\right|_{\boldsymbol{\chi}} + \mathbf{w} \cdot \nabla_{\boldsymbol{\chi}} f \end{aligned} \tag{22}$$

$\mathbf{w} = \dot{\boldsymbol{\psi}}(\mathbf{X}, t) = \dot{\boldsymbol{\chi}}$ is the *particle referential velocity*, that is the velocity of a material point seen from the referential frame. It is easy to verify that

$$\mathbf{v} = \dot{\hat{\boldsymbol{\varphi}}}(\boldsymbol{\chi}, t) = \left.\frac{\partial \hat{\boldsymbol{\varphi}}}{\partial t}\right|_{\boldsymbol{\chi}} + \nabla_{\boldsymbol{\chi}} \hat{\boldsymbol{\varphi}}(\boldsymbol{\chi}, t) \left.\frac{\partial \boldsymbol{\psi}(\mathbf{X}, t)}{\partial t}\right|_{\mathbf{X}} = \hat{\mathbf{v}} + \hat{\mathbf{F}}\mathbf{w} \tag{23}$$

which yields,

$$\mathbf{c} = \mathbf{v} - \hat{\mathbf{v}} = \hat{\mathbf{F}}\mathbf{w} \tag{24}$$

or, in index notation, $c_i = v_i - \hat{v}_i = \hat{F}_{ij} w_j$. The *convective velocity* $\mathbf{c}$ is the velocity of the material relative to the mesh. Using (24), it is possible to cast (22) in a more intuitive way, as

$$\dot{f}(\boldsymbol{\chi}, t) = \left.\frac{\partial f}{\partial t}\right|_{\boldsymbol{\chi}} + \nabla_{\boldsymbol{\chi}} f \cdot (\hat{\mathbf{F}}^{-1}\mathbf{c}) = \left.\frac{\partial f}{\partial t}\right|_{\boldsymbol{\chi}} + \mathbf{c} \cdot \nabla_{\mathbf{x}} f \tag{25}$$

Remarks

(i) In the Lagrangian limit, $\boldsymbol{\chi} \equiv \mathbf{X}$, $\hat{\mathbf{v}} = \mathbf{v}$, and $\hat{\mathbf{F}} \equiv \mathbf{F}$, $\forall t$, so that $\mathbf{w} = \dot{\boldsymbol{\chi}} = \dot{\mathbf{X}} = \mathbf{0}$ and $\mathbf{c} = \mathbf{0}$.
(ii) In the Eulerian limit, $\boldsymbol{\chi} \equiv \mathbf{x}$, $\hat{\mathbf{v}} = \mathbf{0}$, and $\hat{\mathbf{F}} \equiv \mathbf{I}$, $\forall t$, so that $\mathbf{w} = \dot{\boldsymbol{\chi}} = \dot{\mathbf{x}} = \mathbf{v}$ and $\mathbf{c} = \mathbf{I}\mathbf{w} = \mathbf{v}$.

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4. THE ALE EQUATIONS OF COMPRESSIBLE FLOWS

As an alternative to the derivations in [9], a quasi-Eulerian ALE space–time formulation will be presented. The terminology 'quasi-Eulerian' is due to the fact that the equations will be expressed in terms of the current configuration, with the exception of the time derivatives. For this purpose, the generalized Leibniz transport theorem becomes very useful.

4.1. Leibniz transport theorem

The Leibniz transport theorem expresses the time rate of the integral of a scalar f over a control volume Ω whose boundaries move with an arbitrary velocity $\hat{\mathbf{v}}$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega=\hat{\boldsymbol{\varphi}}(\hat{\Omega})} f\,\mathrm{d}\Omega=\int_{\Omega}\hat{J}^{-1}\,\frac{\partial}{\partial t}\bigg|_{\boldsymbol{\chi}}\left(\hat{J}\;f\right)\mathrm{d}\Omega=\int_{\Omega}\frac{\partial f}{\partial t}\bigg|_{\mathbf{x}}\mathrm{d}\Omega+\int_{\Gamma=\partial\Omega} f\hat{\mathbf{v}}\cdot\mathbf{n}\,\mathrm{d}\Gamma \tag{26}$$

A proof is detailed in Appendix A.

4.2. Quasi-Eulerian, integral form of the ALE equations

Let us first apply the balance laws to a control volume Ω whose boundaries are fixed (an Eulerian control volume, with $\hat{\mathbf{v}}=\mathbf{0}$):

$$0=\int_{\Omega}\frac{\partial\rho}{\partial t}\bigg|_{\mathbf{x}}\mathrm{d}\Omega+\int_{\Gamma}\rho\mathbf{v}\cdot\mathbf{n}\,\mathrm{d}\Gamma \tag{27}$$

$$0=\int_{\Omega}\frac{\partial\rho\mathbf{v}}{\partial t}\bigg|_{\mathbf{x}}\mathrm{d}\Omega+\int_{\Gamma}(\rho\mathbf{v}\otimes\mathbf{v}-\boldsymbol{\sigma})\mathbf{n}\,\mathrm{d}\Gamma-\int_{\Omega}\rho\;\mathbf{g}\,\mathrm{d}\Omega \tag{28}$$

$$0=\int_{\Omega}\frac{\partial\rho E}{\partial t}\bigg|_{\mathbf{x}}\mathrm{d}\Omega+\int_{\Gamma}(\rho E\mathbf{v}-\boldsymbol{\sigma}^{\mathrm{T}}\mathbf{v}+\mathbf{q})\cdot\mathbf{n}\,\mathrm{d}\Gamma-\int_{\Omega}\rho(\mathbf{v}\cdot\mathbf{g}+s)\,\mathrm{d}\Omega \tag{29}$$

Here E is the total energy, the sum of the internal energy e and the kinetic energy $\mathbf{v}\cdot\mathbf{v}/2$, $\mathbf{g}$ the body force, and s the heat source. All previous quantities are defined per unit mass. In addition, ρ is the density, $\boldsymbol{\sigma}$ is the stress tensor, and $\mathbf{q}$ is the heat flux. Using (26), it is possible to recast the integrals of the time derivatives in terms of a control volume that coincides with Ω at time t, but moves with an arbitrary velocity $\hat{\mathbf{v}}$. Recalling $\mathbf{c}=\mathbf{v}-\hat{\mathbf{v}}$, it is easily derived:

$$0=\int_{\Omega}\hat{J}^{-1}\,\frac{\partial(\hat{J}\rho)}{\partial t}\bigg|_{\boldsymbol{\chi}}\mathrm{d}\Omega+\int_{\Gamma}\rho\mathbf{c}\cdot\mathbf{n}\,\mathrm{d}\Gamma \tag{30}$$

$$0=\int_{\Omega}\hat{J}^{-1}\,\frac{\partial(\hat{J}\rho\mathbf{v})}{\partial t}\bigg|_{\boldsymbol{\chi}}\mathrm{d}\Omega+\int_{\Gamma}(\rho\mathbf{v}\otimes\mathbf{c}-\boldsymbol{\sigma})\mathbf{n}\,\mathrm{d}\Gamma-\int_{\Omega}\rho\mathbf{g}\,\mathrm{d}\Omega \tag{31}$$

$$0=\int_{\Omega}\hat{J}^{-1}\,\frac{\partial(\hat{J}\rho E)}{\partial t}\bigg|_{\boldsymbol{\chi}}\mathrm{d}\Omega+\int_{\Gamma}(\rho E\mathbf{c}-\boldsymbol{\sigma}^{\mathrm{T}}\mathbf{v}+\mathbf{q})\cdot\mathbf{n}\,\mathrm{d}\Gamma-\int_{\Omega}\rho(\mathbf{v}\cdot\mathbf{g}+s)\,\mathrm{d}\Omega \tag{32}$$

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Applying the divergence theorem in its vector and tensor forms yields

$$\boxed{\hat{J}^{-1}\partial_t|_{\boldsymbol{\chi}}(\hat{J}\mathbf{U}(\mathbf{Y}))+\partial_{x_i}\mathbf{G}_i(\mathbf{Y})+\mathbf{Z}(\mathbf{Y})=\mathbf{0}} \tag{33}$$

where the following definitions apply:

$$\mathbf{U}=\begin{bmatrix}\rho\\ \rho v_1\\ \rho v_2\\ \rho v_3\\ \rho E\end{bmatrix},\quad \mathbf{Z}=\begin{bmatrix}0\\ -\rho g_1\\ -\rho g_2\\ -\rho g_3\\ -\rho v_i g_i-\rho s\end{bmatrix},\quad \mathbf{G}_i=\begin{bmatrix}\rho c_i\\ \rho v_1 c_i-\sigma_{1i}\\ \rho v_2 c_i-\sigma_{2i}\\ \rho v_3 c_i-\sigma_{3i}\\ \rho E c_i-v_j\sigma_{ji}+q_i\end{bmatrix} \tag{34}$$

with $i=1,2,3$.

4.3. Mie–Grüneisen constitutive laws

It is assumed that the materials under consideration do not possess shear strength, so that the Cauchy stress tensor $\boldsymbol{\sigma}$ reduces to an isotropic tensor, dependent only on the thermodynamic pressure

$$\sigma_{ij}=-p\delta_{ij} \tag{35}$$

with δ_{ij}, the Kronecker tensor. Mie–Grüneisen materials satisfy an equation of state of the form $p=f_1(\rho;\rho_{\mathrm{r}},e_{\mathrm{r}})+f_2(\rho;\rho_{\mathrm{r}},e_{\mathrm{r}})e$, where ρ_{r} and e_{r} are fixed reference thermodynamic states. More succinctly,

$$p=f_1(\rho)+f_2(\rho)e \tag{36}$$

Remarks

(i) If $f_1=0$ and $f_2=(\gamma-1)\rho$, the equation of state for an ideal gas, $p=(\gamma-1)\rho e$, is recovered.
(ii) Note that

$$\mathbf{G}_i=c_i\mathbf{U}+\mathbf{G}_i^{\mathrm{L}},\quad \mathbf{G}_i^{\mathrm{L}}=\begin{bmatrix}0\\ -\sigma_{1i}\\ -\sigma_{2i}\\ -\sigma_{3i}\\ q_i-v_j\sigma_{ji}\end{bmatrix}=\begin{bmatrix}0\\ p\delta_{1i}\\ p\delta_{2i}\\ p\delta_{3i}\\ q_i+v_i p\end{bmatrix} \tag{37}$$

where $\mathbf{G}_i^{\mathrm{L}}$ is the Lagrangian limit of the Euler flux Jacobians, as $\mathbf{c}\to\mathbf{0}$.

Thanks to the Mie–Grüneisen constitutive equations, a quasi-linear form of (33) can be derived, namely,

$$\boxed{\mathbf{A}_0\partial_t|_{\boldsymbol{\chi}}\mathbf{Y}+\mathbf{A}_i(\mathbf{Y})\partial_{x_i}\mathbf{Y}+\mathbf{C}(\mathbf{Y})\mathbf{Y}=\mathbf{0}} \tag{38}$$

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The matrices $\mathbf{A}_0$, $\mathbf{A}_i$, and $\mathbf{C}$ (to be given in Section 6) depend on the choice of the solution vector $\mathbf{Y}$.

4.4. A space–time variational formulation

In order to lay the foundations for the subsequent discussion, a space–time variational formulation is presented. The analysis of Galilean invariance is not strictly dependent on the variational formulation adopted, and, for example, similar conclusions hold for alternative space–time or semi-discrete formulations. Given a partition $0=t_0<t_1<t_2<\cdots<t_{N-1}<t_N=T$ of the time interval $I=]0,T]$, let $I_n=]t_n,t_{n+1}]$, so that $]0,T]=\bigcup_{n=0}^{N-1} I_n$. The space–time domain $\hat{Q}=\hat{\Omega}\times I$ can be divided into time slabs

$$\hat{Q}_n=\hat{\Omega}\times I_n \tag{39}$$

with 'lateral' boundary $\hat{P}_n=\hat{\Gamma}\times I_n$ ($\hat{\Gamma}=\partial\hat{\Omega}$ is the boundary of $\hat{\Omega}$, see Figure 4). We will only make use of discretizations *prismatic* in time, with respect to the referential frame. Namely, the material domain $\hat{\Omega}$ is divided into subdomains $\hat{\Omega}^e$ (elements in space, a partition of the initial configuration). Thus $\hat{\Omega}=\overline{\bigcup_{e=1}^{n_{\mathrm{el}}}\hat{\Omega}^e}$, and, consequently, a typical space–time element is given by the *referential* prism (i.e. tensor product domain)

$$\hat{Q}_n^e=\hat{\Omega}^e\times I_n \tag{40}$$

It is also assumed that the space–time boundary is partitioned as $\hat{P}_n=\hat{P}_n^g\cup\hat{P}_n^h$, $\hat{P}_n^g\cap\hat{P}_n^h=\emptyset$ (i.e. $\hat{P}_n$ is divided into a Dirichlet boundary $\hat{P}_n^g$ and a Neumann boundary $\hat{P}_n^h$). As shown in

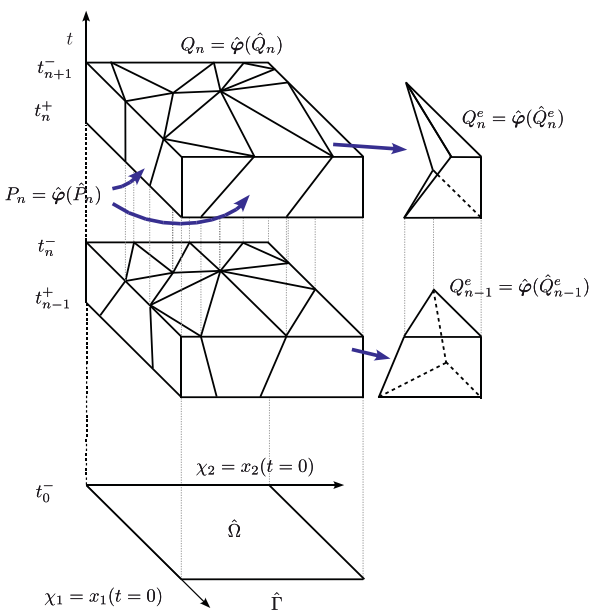

Figure 4. General finite element discretization in space–time.

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Figure 4, the previous definitions can be *pushed forward* to the current configuration, for which the following expressions hold:

$$Q_n \overset{\text{def}}{=} \Omega \times I_n = \hat{\boldsymbol{\varphi}}(\hat{Q}_n) \tag{41}$$

$$P_n = \hat{\boldsymbol{\varphi}}(\hat{P}_n) \tag{42}$$

$$\Gamma_n = \hat{\boldsymbol{\varphi}}(\hat{\Gamma}, t_n) \tag{43}$$

$$\Omega_n^e = \hat{\boldsymbol{\varphi}}(\hat{\Omega}^e, t_n) \tag{44}$$

$$Q_n^e \overset{\text{def}}{=} \Omega^e \times I_n = \hat{\boldsymbol{\varphi}}(\hat{Q}_n^e) \tag{45}$$

$$P_n^e = \hat{\boldsymbol{\varphi}}(\hat{P}_n^e) \tag{46}$$

It should be obvious that, due to the structure of the mapping $\hat{\boldsymbol{\varphi}}$, $\Omega_n \times I_n \neq \Omega \times I_n$, and $\Omega_n^e \times I_n \neq \Omega^e \times I_n$. The test and trial function spaces can therefore be defined as

$$\mathscr{S}_n^h = \{\mathbf{V}^h : \mathbf{V}^h \in (C^0(Q))^m, \mathbf{V}^h|_{Q_n^e} \in (\mathscr{P}_1(Q_n^e))^m, \mathbf{V}^h = \mathbf{G}_{bc}(t) \text{ on } P_n^g\} \tag{47}$$

$$\mathscr{V}_n^h = \{\mathbf{W}^h : \mathbf{W}^h \in (C^0(\Omega))^m, \mathbf{W}^h|_{Q_n^e} \in (\mathscr{P}_1(\Omega^e) \times \mathscr{P}_0(I_n))^m, \mathbf{W}^h = \mathbf{0} \text{ on } P_n^g\} \tag{48}$$

where $\mathbf{G}_{bc}(t)$ is the vector of Dirichlet boundary conditions, $\mathscr{P}_k$ is the set of polynomials up to degree k, and $m = n_d + 2$, $n_d = 1$, 2, or 3. The trial function space $\mathscr{S}_n^h$ is given by the piecewise-linear, continuous functions on $Q = \hat{\boldsymbol{\varphi}}(\hat{Q})$, while the test function space $\mathscr{V}_n^h$ is given by functions that are continuous piecewise-linear in space and discontinuous, piecewise-constant in time. The variational statement reads as follows.

Find $\mathbf{Y}^h \in \mathscr{S}_n^h$, such that, $\forall \mathbf{W}^h \in \mathscr{V}_n^h$,

$$\begin{aligned} 0 = &\int_{\Omega(t_{n+1})} \mathbf{W}^h(\boldsymbol{\chi}) \cdot \mathbf{U}(\mathbf{Y}^h(\mathbf{x}, t_{n+1})) - \int_{\Omega(t_n)} \mathbf{W}^h(\boldsymbol{\chi}) \cdot \mathbf{U}(\mathbf{Y}^h(\mathbf{x}, t_n)) \, \mathrm{d}\Omega \\ &- \int_{Q_n} \mathbf{W}^h_{,i} \cdot \mathbf{G}_i(\mathbf{Y}^h) \, \mathrm{d}Q + \int_{Q_n} \mathbf{W}^h \cdot \mathbf{Z}(\mathbf{Y}^h) \, \mathrm{d}Q \\ &+ \int_{P_n^g} \mathbf{W}^h \cdot \mathbf{G}_i(\mathbf{Y}^h) n_i \, \mathrm{d}P + \int_{P_n^h} \mathbf{W}^h \cdot \mathbf{H}_i n_i \, \mathrm{d}P \\ &+ \mathsf{SUPG}(\mathbf{W}^h, \mathbf{Y}^h) + \mathsf{DC}(\mathbf{W}^h, \mathbf{Y}^h) \end{aligned} \tag{49}$$

where n_i is the ith component of the normal to the space–time boundary, and $\mathbf{H}_i$ is the Neumann flux across the boundary in the ith direction. The SUPG operator $\mathsf{SUPG}(\mathbf{W}^h, \mathbf{Y}^h)$ will be defined subsequently. Away from discontinuities, where the flow field is smooth, $\mathsf{DC}(\mathbf{W}^h, \mathbf{Y}^h)$ vanishes, and for this reason, it will be omitted in the following discussion. The proposed formulation is second-order-in-time and, following derivations analogous to [20], it can be easily proven to embed global conservation of mass, momentum, and total energy.

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4.5. SUPG Stabilization

The SUPG stabilization operator can be abstractly defined as

$$\mathsf{SUPG}(\mathbf{W}^h, \mathbf{Y}^h) = -\sum_{e=1}^{(n_{\mathrm{el}})_n} \int_{Q_n^e} (\mathbf{L}_{\mathrm{SH}}^* \mathbf{W}_h) \cdot \boldsymbol{\tau}\, \mathbf{Res}(\mathbf{Y}^h)\, \mathrm{d}Q \tag{50}$$

where

$$\mathbf{Res} = \mathbf{A}_0 \partial_t|_{\boldsymbol{\chi}} + \mathbf{A}_i \partial_{x_i} + \mathbf{C} = \mathbf{L} \tag{51}$$

$$\mathbf{L}_{\mathrm{SH}} = \mathbf{A}_0 \partial_t|_{\boldsymbol{\chi}} + \mathbf{A}_i \partial_{x_i} \tag{52}$$

$$\mathbf{L}_{\mathrm{SH}}^* = -\mathbf{A}_0^{\mathrm{T}} \partial_t|_{\boldsymbol{\chi}} - \mathbf{A}_i^{\mathrm{T}} \partial_{x_i} \tag{53}$$

$$\boldsymbol{\tau} = \boldsymbol{\tau}(\Delta t, h_e, \mathbf{A}_0, \mathbf{A}_i, \mathbf{C}, \ldots) \tag{54}$$

Δt is the time increment, and h_e is the eth element mesh scale. In the discussion that follows, a precise definition of $\boldsymbol{\tau}$ is not needed. Instead, its functional dependence on the parameters and various terms in the formulation are sufficient to fully understand the issues under investigation.

Remarks

(i) The expression for $\mathbf{Res}(\mathbf{Y}^h)$ is obtained by applying the quasi-linear form (38) of the Galerkin residual (33). Hence higher-order inconsistency in the Galielan invariance properties of the residual are to be expected. Numerical evidence [16] shows that, usually, such inconsistencies are not critical.

(ii) Most importantly, a consistently stabilized method *must* ensure strict Galilean invariance for the perturbation to the Galerkin test function $(\mathbf{L}_{\mathrm{SH}}^* \mathbf{W}_h) \cdot \boldsymbol{\tau}$. If this is not the case, the stability properties of the overall formulation depend on the *observer*, clearly a paradox. The discussion that follows is aimed to investigate in detail this particular issue.

5. GALILEAN INVARIANCE AND RESIDUAL STRUCTURE

Before undertaking an exhaustive discussion on the construction of the SUPG operator, it is important to understand how the numerical Galerkin residuals transform. It will be shown that for some choices of the solution variables, it is possible to maintain invariance properties if the numerical residuals are in advective form, independently of the integration quadrature adopted. This result *does not* hold for *any* solution vector, as was clearly derived in [9], for the case of conservation variables. Therefore, a key point to be made is the following: *not all forms of the numerical, non-vanishing residuals transform correctly*.

Using the following definitions of the advective form of the mass, momentum, and internal energy residuals,

$$\mathsf{Res}^{\rho}(\rho; \mathbf{c}, \mathbf{v}, t) = \left.\frac{\partial \rho}{\partial t}\right|_{\boldsymbol{\chi}} + c_j \frac{\partial \rho}{\partial x_j} + \rho \frac{\partial v_j}{\partial x_j} \tag{55}$$

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$$\mathsf{Res}_i^{\mathbf{v}}(\rho,\, p;\, \mathbf{c},\, \mathbf{v},\, t) = \rho\, \frac{\partial v_i}{\partial t}\bigg|_{\boldsymbol{\chi}} + \rho c_j\, \frac{\partial v_i}{\partial x_j} + \frac{\partial p}{\partial x_i} - \rho g_i \tag{56}$$

$$\mathsf{Res}^{e}(\rho,\, e,\, p;\, \mathbf{c},\, \mathbf{v},\, t) = \rho\, \frac{\partial e}{\partial t}\bigg|_{\boldsymbol{\chi}} + \rho c_j \frac{\partial e}{\partial x_j} + \frac{\partial v_j}{\partial x_j} p - \rho s \tag{57}$$

it easy to prove that (33), valid for a smooth flow, can be expressed as

$$0 = \mathsf{Res}^{\rho}(\rho;\, \mathbf{c},\, \mathbf{v},\, t) \tag{58}$$

$$\begin{aligned} 0 &= \mathsf{Res}_i^{\rho\mathbf{v}}(\rho,\, p;\, \mathbf{c},\, \mathbf{v},\, t) \\ &= v_i \mathsf{Res}^{\rho}(\rho;\, \mathbf{c},\, \mathbf{v},\, t) + \mathsf{Res}_i^{\mathbf{v}}(\rho,\, p;\, \mathbf{c},\, \mathbf{v},\, t) \end{aligned} \tag{59}$$

$$\begin{aligned} 0 &= \mathsf{Res}^{E}(\rho,\, e,\, p;\, \mathbf{c},\, \mathbf{v},\, t) \\ &= \left(e + \frac{v_k v_k}{2}\right) \mathsf{Res}^{\rho}(\rho;\, \mathbf{c},\, \mathbf{v},\, t) \\ &\quad + v_i \mathsf{Res}_i^{\mathbf{v}}(\rho,\, p;\, \mathbf{c},\, \mathbf{v},\, t) + \mathsf{Res}^{e}(\rho,\, e,\, p;\, \mathbf{c},\, \mathbf{v},\, t) \end{aligned} \tag{60}$$

Recalling that $\mathbf{x}(\mathbf{X},\, t=0) = \boldsymbol{\chi}(\mathbf{X},\, t=0) = \mathbf{X}$, it easy to verify that a Galilean transformation yields,

$$\begin{bmatrix} \tilde{t} \\ \tilde{\mathbf{c}} \\ \tilde{\mathbf{v}} \end{bmatrix} = \begin{bmatrix} t \\ \mathbf{c} \\ \mathbf{v} - \mathbf{V}^{\mathrm{G}} \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} t \\ \mathbf{c} \\ \mathbf{v} \end{bmatrix} = \begin{bmatrix} \tilde{t} \\ \tilde{\mathbf{c}} \\ \tilde{\mathbf{v}} + \mathbf{V}^{\mathrm{G}} \end{bmatrix} \tag{61}$$

Hence,

$$\mathsf{Res}^{\rho}(\rho;\, \mathbf{v},\, \mathbf{c},\, t) \xrightarrow{\mathrm{G}} \mathsf{Res}^{\rho}(\rho;\, \tilde{\mathbf{v}},\, \tilde{\mathbf{c}},\, \tilde{t}) \tag{62}$$

$$\mathsf{Res}_i^{\mathbf{v}}(\rho,\, p;\, \mathbf{v},\, \mathbf{c},\, t) \xrightarrow{\mathrm{G}} \mathsf{Res}_i^{\tilde{\mathbf{v}}}(\rho,\, p;\, \tilde{\mathbf{v}},\, \tilde{\mathbf{c}},\, \tilde{t}) \tag{63}$$

$$\mathsf{Res}^{e}(\rho,\, e,\, p;\, \mathbf{v},\, \mathbf{c},\, t) \xrightarrow{\mathrm{G}} \mathsf{Res}^{e}(\rho,\, e,\, p;\, \tilde{\mathbf{v}},\, \tilde{\mathbf{c}},\, \tilde{t}) \tag{64}$$

$$\mathsf{Res}_i^{\rho\mathbf{v}}(\rho,\, p;\, \mathbf{v},\, \mathbf{c},\, t) \xrightarrow{\mathrm{G}} \mathsf{Res}_i^{\rho\tilde{\mathbf{v}}}(\rho,\, p;\, \tilde{\mathbf{v}},\, \tilde{\mathbf{c}},\, \tilde{t}) + V_i^{\mathrm{G}} \mathsf{Res}^{\rho}(\rho;\, \tilde{\mathbf{v}},\, \tilde{\mathbf{c}},\, \tilde{t}) \tag{65}$$

$$\begin{aligned} \mathsf{Res}^{\mathrm{E}}(\rho,\, e,\, p;\, \mathbf{v},\, \mathbf{c},\, t) &\xrightarrow{\mathrm{G}} \mathsf{Res}^{\mathrm{E}}(\rho,\, e,\, p;\, \tilde{\mathbf{v}},\, \tilde{\mathbf{c}},\, \tilde{t}) \\ &\quad + V_i^{\mathrm{G}} \mathsf{Res}_i^{\rho\tilde{\mathbf{v}}}(\rho,\, p;\, \tilde{\mathbf{v}},\, \tilde{\mathbf{c}},\, \tilde{t}) + \frac{V_k^{\mathrm{G}} V_k^{\mathrm{G}}}{2} \mathsf{Res}^{\rho}(\rho;\, \tilde{\mathbf{v}},\, \tilde{\mathbf{c}},\, \tilde{t}) \end{aligned} \tag{66}$$

Hence, the equations would transform appropriately, if the terms multiplied by the transformation velocity $\mathbf{V}^{\mathrm{G}}$ annihilated exactly. In other words, if an exact multiscale decomposition of the solution were available, the resulting equations would satisfy the invariance principle.

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If, however, as already mentioned in [9, 16], the subgrid-scale problem is solved only approximately, the situation is different and the numerical residuals are not necessarily invariant.

No matter the numerical quadrature used, the numerical approximations to the 'advective' residuals $\mathsf{Res}^{h;\rho}$, $\mathsf{Res}^{h;\mathbf{v}}$, and $\mathsf{Res}^{h;e}$ would transform correctly, if, for example, the set of solution variables is given by $[\rho,\ \mathbf{v}^{\mathrm{T}},\ p]^{\mathrm{T}}$, $[\rho,\ \mathbf{v}^{\mathrm{T}},\ e]^{\mathrm{T}}$, or $[e,\ \mathbf{v}^{\mathrm{T}},\ p]^{\mathrm{T}}$. More generally, if $\mathbf{v}$ is a variable in the solution vector, and the remaining two entries are given by functions of the thermodynamic quantities, the resulting advective form of the residuals would transform correctly. This is due to the fact that the advective form of the residual for the previous set of solution variables contains the velocity $\mathbf{v}$ *only* in differentiated form. Instead, $\mathsf{Res}^{h;\rho\mathbf{v}}$ and $\mathsf{Res}^{h;\mathrm{E}}$ would not transform correctly, because $\mathbf{V}^{\mathrm{G}}$ multiplies some of the non-vanishing residual terms.

It is also important to realize, however, that residuals are usually higher-order corrections: In the computations performed in [20, 16], virtually no difference in the results was observed between SUPG operators with and without invariant residuals. In fact, instabilities were experienced only for a non-invariant SUPG test function perturbation, indicating that this case is, by far, the more critical.

6. QUASI-LINEAR FORMS AND INVARIANCE

The quasi-linear differential form of the ALE equations has a central role in the design of SUPG operators, which make use of a fairly arbitrary combination of its Euler flux Jacobians to define a perturbation to the Bubnov–Galerkin test function space. Hence, a practical requirement to be enforced is that *every* Euler flux Jacobian must be invariant, or one cannot expect the perturbed test space to be independent of the observer.

6.1. Pressure primitive variables

The case of pressure primitive variables ($\mathbf{Y} = [e\ \mathbf{v}^{\mathrm{T}}\ p]^{\mathrm{T}}$) is now analysed. The heat flux $\mathbf{q}$ is assumed absent. Pressure primitive variables are of great interest in the aerospace community, since they allow to span the compressible and incompressible limits of the Euler equations. For the sake of completeness, Appendix B presents the case of the density–pressure variables ($\mathbf{Y} = [\rho\ \mathbf{v}^{\mathrm{T}}\ p]^{\mathrm{T}}$), and density-internal energy variables ($\mathbf{Y} = [\rho\ \mathbf{v}^{\mathrm{T}}\ e]^{\mathrm{T}}$), the latter being traditionally used in shock hydrodynamics computations. It will be shown how to successfully address the issue of lack of Galilean invariance affecting classical SUPG formulations, by using the advective form of the residuals when constructing the Euler flux Jacobians needed in the definition of the SUPG operator. In the literature of formulations for the pressure primitive variables [21–23], the temperature T is typically used in place of the internal energy e. However, the familiar expressions for the Euler Jacobians can be recovered noticing that $e = c_v T$, where c_v is the specific heat for an isocoric thermodynamic transformation. Recalling that $\partial_t|_{\boldsymbol{\chi}} \hat{J} = \hat{J}\ \boldsymbol{\nabla}_{\mathbf{x}} \cdot \hat{\mathbf{v}}$ (for a proof, see (A3) in Appendix A), the following manipulations will prove useful:

$$
\begin{aligned}
\mathbf{0} &= \hat{J}^{-1}\partial_t|_{\boldsymbol{\chi}}(\hat{J}\mathbf{U}) + \partial_{x_i}\mathbf{G}_i + \mathbf{Z} \\
&= \partial_t|_{\boldsymbol{\chi}}\mathbf{U} + \mathbf{U}\boldsymbol{\nabla}_{\mathbf{x}} \cdot \hat{\mathbf{v}} + \partial_{x_i}(c_i\mathbf{U} + \mathbf{G}_i^{\mathrm{L}}) + \mathbf{Z} \\
&= \partial_t|_{\boldsymbol{\chi}}\mathbf{U} + \mathbf{U}\boldsymbol{\nabla}_{\mathbf{x}} \cdot \hat{\mathbf{v}} + \mathbf{U}\boldsymbol{\nabla}_{\mathbf{x}} \cdot \mathbf{c} + c_i\partial_{x_i}\mathbf{U} + \partial_{x_i}\mathbf{G}_i^{\mathrm{L}} + \mathbf{Z} \\
&= \partial_t|_{\boldsymbol{\chi}}\mathbf{U} + c_i\partial_{x_i}\mathbf{U} + \mathbf{U}\boldsymbol{\nabla}_{\mathbf{x}} \cdot \mathbf{v} + \partial_{x_i}\mathbf{G}_i^{\mathrm{L}} + \mathbf{Z}
\end{aligned}
\tag{67}
$$

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6.1.1. The 'standard', non-invariant approach. Starting from (67), it easy to derive

$$\partial_\bullet \mathbf{U} = \partial_\bullet \begin{bmatrix} \rho \\ \rho v_1 \\ \rho v_2 \\ \rho v_3 \\ \rho E \end{bmatrix} = \partial_\bullet \left(\rho \begin{bmatrix} 1 \\ v_1 \\ v_2 \\ v_3 \\ E \end{bmatrix} \right) = \rho \begin{bmatrix} 0 \\ \partial_\bullet v_1 \\ \partial_\bullet v_2 \\ \partial_\bullet v_3 \\ v_k \partial_\bullet v_k + \partial_\bullet e \end{bmatrix} + \partial_\bullet \rho \begin{bmatrix} 1 \\ v_1 \\ v_2 \\ v_3 \\ E \end{bmatrix} \tag{68}$$

where either $\partial_\bullet = \partial_t|_{\boldsymbol{\chi}}$ or $\partial_\bullet = \partial_{x_j}$, and $\partial_\bullet \rho = \rho_{,e} \partial_\bullet e + \rho_{,p} \partial_\bullet p$, with $\rho_{,e} = \partial\rho/\partial e|_p$, and $\rho_{,p} = \partial\rho/\partial p|_e$. The quasi-linear vector form (38) would then incorporate the following definitions:

$$\mathbf{A}_0^{(NG)} = \begin{bmatrix} \rho_{,e} & 0 & 0 & 0 & \rho_{,p} \\ v_1 \rho_{,e} & \rho & 0 & 0 & v_1 \rho_{,p} \\ v_2 \rho_{,e} & 0 & \rho & 0 & v_2 \rho_{,p} \\ v_3 \rho_{,e} & 0 & 0 & \rho & v_3 \rho_{,p} \\ E\rho_{,e} + \rho & \rho v_1 & \rho v_2 & \rho v_3 & E\rho_{,p} \end{bmatrix} \tag{69}$$

$$\mathbf{C}^{(NG)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -g_1 \tilde{f}_\rho & 0 & 0 & 0 & 0 \\ -g_2 \tilde{f}_\rho & 0 & 0 & 0 & 0 \\ -g_3 \tilde{f}_\rho & 0 & 0 & 0 & 0 \\ -s \tilde{f}_\rho & -\rho g_1 & -\rho g_2 & -\rho g_3 & 0 \end{bmatrix} \tag{70}$$

where $\tilde{f}_\rho = \rho(e, p)/e$, and, for $i = 1, 2, 3$,

$$\mathbf{A}_i^{(NG)} = \left[\begin{array}{c|c|c|c|c} c_i \rho_{,e} & \rho\delta_{1i} & \rho\delta_{2i} & \rho\delta_{3i} & c_i \rho_{,p} \\ \hline c_i v_1 \rho_{,e} & \rho c_i + \rho v_1 \delta_{1i} & \rho v_1 \delta_{2i} & \rho v_1 \delta_{3i} & c_i v_1 \rho_{,p} + \delta_{1i} \\ \hline c_i v_2 \rho_{,e} & \rho v_2 \delta_{1i} & \rho c_i + \rho v_2 \delta_{2i} & \rho v_2 \delta_{3i} & c_i v_2 \rho_{,p} + \delta_{2i} \\ \hline c_i v_3 \rho_{,e} & \rho v_3 \delta_{1i} & \rho v_3 \delta_{2i} & \rho c_i + \rho v_3 \delta_{3i} & c_i v_3 \rho_{,p} + \delta_{3i} \\ \hline c_i(\rho + E\rho_{,e}) & \rho c_i v_1 + & \rho c_i v_2 + & \rho c_i v_3 + & E c_i \rho_{,p} + v_i \\ & (\rho E + p)\delta_{1i} & (\rho E + p)\delta_{2i} & (\rho E + p)\delta_{3i} & \end{array}\right] \tag{71}$$

Remarks

(i) This choice leads to Jacobians of the Euler fluxes which are *not* invariant if considered separately. By inspection, it is easy to realize that there is a large number of terms which

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contain components of the velocity vector $\mathbf{v}$. Therefore, a single Euler flux Jacobian or an arbitrary combination of Euler flux Jacobians are not necessarily invariant. This is precisely the situation for the perturbation to the test function $-(\mathbf{L}_{\mathrm{SH}}^{*}\mathbf{W}^{h})\cdot\boldsymbol{\tau} = (\mathbf{A}_0^{\mathrm{T}}\,\partial_t\big|_{\boldsymbol{\chi}}\mathbf{W}^{h} + \mathbf{A}_i^{\mathrm{T}}\partial_{x_i}\mathbf{W}^{h})\boldsymbol{\tau}$, which lacks invariance properties, with potentially very negative consequences on the overall stability of the formulation.

(ii) Although in principle it is possible to develop a tensor $\boldsymbol{\tau}$ producing an invariant perturbation of the test function, in practice, the current structure of the Jacobians makes this task extremely difficult.

(iii) By taking the Eulerian limit ($\hat{\mathbf{v}} \to \mathbf{0}$ and $\mathbf{c} \to \mathbf{v}$), one can easily obtain the form of the Euler flux Jacobians commonly implemented in stabilized finite element methods (see, e.g. [21]), which are therefore non-invariant.

(iv) As a justification for the invariance inconsistencies found in the literature of stabilized methods, it is worth noting that it is virtually impossible to discern whether a velocity term transforms correctly, if only the Eulerian form of the equations is available, for which $\mathbf{c} = \mathbf{v}$. Hence, the reverse approach is needed, in which first a consistent ALE formulation is developed, and then the Eulerian equations are derived as a limit.

6.1.2. Galilean invariant approach. As already noticed, the momentum equation (59) contains its advective form (56), and, in addition, the product of the velocity components v_i times the mass conservation equation (58). Analogously, the total energy equation (60) contains the sum of the internal energy equation (57) and the kinetic energy equation, that is, the scalar product of the momentum equation (59) and the velocity $\mathbf{v}$. Therefore, if the advective forms (55)–(57) are used, the following alternative form of the Euler Jacobians can be derived:

$$\mathbf{A}_0^{(\mathrm{Gal})} = \begin{bmatrix} \rho_{,e} & 0 & 0 & 0 & \rho_{,p} \\ 0 & \rho & 0 & 0 & 0 \\ 0 & 0 & \rho & 0 & 0 \\ 0 & 0 & 0 & \rho & 0 \\ \rho & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{C}^{(\mathrm{Gal})} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -g_1\tilde{f}_\rho & 0 & 0 & 0 & 0 \\ -g_2\tilde{f}_\rho & 0 & 0 & 0 & 0 \\ -g_3\tilde{f}_\rho & 0 & 0 & 0 & 0 \\ -s\tilde{f}_\rho & 0 & 0 & 0 & 0 \end{bmatrix} \tag{72}$$

and, for $i = 1, 2, 3$,

$$\mathbf{A}_i^{(\mathrm{Gal})} = \left[\begin{array}{c|c|c|c|c} c_i\rho_{,e} & \rho\,\delta_{1i} & \rho\,\delta_{2i} & \rho\,\delta_{3i} & c_i\rho_{,p} \\ \hline 0 & \rho c_i & 0 & 0 & \delta_{1i} \\ \hline 0 & 0 & \rho c_i & 0 & \delta_{2i} \\ \hline 0 & 0 & 0 & \rho c_i & \delta_{3i} \\ \hline \rho c_i & p\,\delta_{1i} & p\,\delta_{2i} & p\,\delta_{3i} & 0 \end{array}\right] \tag{73}$$

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Remarks

(i) Each of the generalized advective matrices developed respects the principle of Galilean invariance, since they are function of $\mathbf{c}$, p, ρ, $e_{,p}$, and $e_{,\rho}$, all invariant quantities.
(ii) One can think about the proposed approach as being 'minimalist'. In fact, it produces the minimal number of entries in the Jacobians while still retaining the generalized advective structure of the quasi-linear form, now reduced to the mass conservation equation, the advective form of the momentum equation, and the advective form of the internal energy equation.
(iii) By comparison with the standard Jacobians (70)–(71), the Galilean invariant Jacobians (72)–(73) require much fewer terms to be computed.
(iv) As documented in Appendix B, similar conclusions can be drawn in the case density–pressure, and density-internal energy variables.
(v) The proposed approach works because this specific choice of variables yields a point-wise invariant expression of the advective form of the compressible Euler equations. For other sets of variables, such as conservation or entropy variables, the advective form does not help removing Galilean inconsistencies, as explained in detail in [9]. An effective way to construct invariant SUPG operators for conservation or entropy variables is still to be found.

7. CONCLUDING REMARKS

The present article complements previous work in [9], by extending the analysis to formulations written in terms of the current configuration reference frame, more commonly used in the computational fluid dynamics community. Once more, it was shown that most of the stabilization operators designed to date for compressible flow applications on Eulerian meshes *do not* satisfy the principle of Galilean invariance. As documented in [16] for the Lagrangian limit, the price to be paid for such flaw can be significant, since a non-invariant Petrov–Galerkin test space directly affects the stability properties of the underlying stabilized method. Due to its significantly reduced computational cost, the new approach has potential in the context of complex-geometry, multi-physics applications.

APPENDIX A: PROOF OF THE LEIBNIZ TRANSPORT THEOREM

Theorem 1
Let Ω be an arbitrary control volume Ω, with boundary Γ. Let $\hat{\mathbf{v}}$ be the velocity with which Γ deforms, and f a scalar. Then, the following formula holds:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega=\hat{\boldsymbol{\varphi}}(\hat{\Omega})} f\,\mathrm{d}\Omega = \int_{\Omega} \left.\frac{\partial f}{\partial t}\right|_{\mathbf{x}} \mathrm{d}\Omega + \int_{\Gamma=\partial\Omega} f\hat{\mathbf{v}}\cdot\mathbf{n}\,\mathrm{d}\Gamma \tag{A1}$$

Proof
Due to the map $\hat{\boldsymbol{\varphi}}$, $\Omega=\hat{\boldsymbol{\varphi}}(\hat{\Omega})$ deforms with velocity $\hat{\mathbf{v}}$. Defining $\hat{\Omega}=\Omega(t=0)$ (i.e. $\boldsymbol{\varphi}(\boldsymbol{\chi}, t=0)=\mathbf{I}$), one has that $\hat{\Omega}$ does not depend on time, and

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega=\hat{\boldsymbol{\varphi}}(\hat{\Omega})} f\,\mathrm{d}\Omega = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\hat{\Omega}} f\hat{J}\,\mathrm{d}\hat{\Omega}$$

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$$= \int_{\hat{\Omega}} \left.\frac{\partial (f\hat{J})}{\partial t}\right|_{\chi} \mathrm{d}\hat{\Omega} = \int_{\hat{\Omega}} \left.\frac{\partial f}{\partial t}\right|_{\chi} \hat{J} + f \left.\frac{\partial \hat{J}}{\partial t}\right|_{\chi} \mathrm{d}\hat{\Omega}$$

$$= \int_{\hat{\Omega}} \left.\frac{\partial f}{\partial t}\right|_{\chi} \hat{J} + f\hat{J}\nabla_{\mathbf{x}} \cdot \hat{\mathbf{v}}\, \mathrm{d}\hat{\Omega}$$

$$= \int_{\hat{\Omega}} \left(\left.\frac{\partial f}{\partial t}\right|_{\chi} - \hat{\mathbf{v}} \cdot \nabla_{\mathbf{x}} f + \nabla_{\mathbf{x}} \cdot (f\hat{\mathbf{v}}) \right) \hat{J}\, \mathrm{d}\hat{\Omega} \tag{A2}$$

where $\nabla_{\mathbf{x}}$ and $\nabla_{\mathbf{x}}\cdot$ are the current configuration (Eulerian) gradient and divergence, and $\mathrm{d}\Omega = \hat{J}\,\mathrm{d}\hat{\Omega}$. In particular, the following identity has been used:

$$\left.\frac{\partial \hat{J}}{\partial t}\right|_{\chi} = \mathbf{D}_{\hat{\mathbf{F}}}(\hat{J})^{\mathrm{T}} : \left(\left.\frac{\partial \hat{\mathbf{F}}}{\partial t}\right|_{\chi} \right)$$

$$= \hat{J}\,\mathrm{tr}\left(\left.\frac{\partial \hat{\mathbf{F}}}{\partial t}\right|_{\chi} \hat{\mathbf{F}}^{-1} \right)$$

$$= \hat{J}\,\mathrm{tr}\left(\left(\left.\frac{\partial}{\partial t}\right|_{\chi} \left(\frac{\partial x_i}{\partial \chi_j} \right) \right) \hat{\mathbf{F}}^{-1} \right)$$

$$= \hat{J}\,\mathrm{tr}\left(\left(\frac{\partial}{\partial \chi_j} \left(\left.\frac{\partial x_i}{\partial t}\right|_{\chi} \right) \right) \hat{\mathbf{F}}^{-1} \right)$$

$$= \hat{J}\,\mathrm{tr}\left(\frac{\partial \hat{v}_i}{\partial \chi_j} \frac{\partial \chi_j}{\partial x_k} \right)$$

$$= \hat{J}\nabla_{\mathbf{x}} \cdot \hat{\mathbf{v}} \tag{A3}$$

$\mathbf{D}_{\hat{\mathbf{F}}}(\cdot)$ is the differentiation operator with respect to the entries of $\hat{\mathbf{F}}$, that is,

$$(D_{\hat{\mathbf{F}}})_{ij}(\cdot) = \frac{\partial(\cdot)}{\partial \hat{F}_{ij}} \tag{A4}$$

Recalling (24), and using (25) with $\hat{\mathbf{v}}$ arbitrary and $\hat{\mathbf{v}} = \mathbf{0}$, it is easily derived:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \left.\frac{\partial f}{\partial t}\right|_{\chi} + (\mathbf{v} - \hat{\mathbf{v}}) \cdot \nabla_{\mathbf{x}} f = \left.\frac{\partial f}{\partial t}\right|_{\mathbf{x}} + \mathbf{v} \cdot \nabla_{\mathbf{x}} f \tag{A5}$$

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or,

$$\left.\frac{\partial f}{\partial t}\right|_{\boldsymbol{\chi}} - \hat{\mathbf{v}} \cdot \boldsymbol{\nabla}_{\mathbf{x}} f = \left.\frac{\partial f}{\partial t}\right|_{\mathbf{x}} \tag{A6}$$

Using the Gauss divergence theorem, (A2) becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega} f\hat{J}\,\mathrm{d}\hat{\Omega} &= \int_{\Omega} \left.\frac{\partial f}{\partial t}\right|_{\mathbf{x}} \mathrm{d}\Omega + \int_{\Omega} \boldsymbol{\nabla}_{\mathbf{x}} \cdot (f\hat{\mathbf{v}})\,\mathrm{d}\Omega \\ &= \int_{\Omega} \left.\frac{\partial f}{\partial t}\right|_{\mathbf{x}} \mathrm{d}\Omega + \int_{\Gamma=\partial\Omega} f\hat{\mathbf{v}} \cdot \mathbf{n}\,\mathrm{d}\Gamma \end{aligned} \tag{A7}$$

which concludes the proof. □

APPENDIX B: QUASI-LINEAR FORMS AND INVARIANCE: ADDITIONAL SETS OF SOLUTION VARIABLES

B.1. Density–pressure variables

In the case of pressure variables ($\mathbf{Y} = [\rho\ \mathbf{v}^{\mathrm{T}}\ p]^{\mathrm{T}}$), the following identity will become very useful:

$$\begin{aligned} \left.\frac{\partial e}{\partial t}\right|_{\boldsymbol{\chi}} + c_j \frac{\partial e}{\partial x_j} &= e_{,p}\left(\left.\frac{\partial p}{\partial t}\right|_{\boldsymbol{\chi}} + c_j \frac{\partial p}{\partial x_j}\right) + e_{,\rho}\left(\left.\frac{\partial \rho}{\partial t}\right|_{\boldsymbol{\chi}} + c_j \frac{\partial \rho}{\partial x_j}\right) \\ &= e_{,p}\left(\left.\frac{\partial p}{\partial t}\right|_{\boldsymbol{\chi}} + c_j \frac{\partial p}{\partial x_j}\right) - e_{,\rho}\rho\,\frac{\partial v_j}{\partial x_j} \end{aligned} \tag{B1}$$

with $e_{,p} = \partial e/\partial p|_{\rho}$ and $e_{,\rho} = \partial e/\partial \rho|_{p}$.

B.1.1. The 'standard', non-invariant approach. Using the traditional Fréchet differentiation of (58)–(60), the quasi-linear vector form reads

$$\mathbf{A}_0(\mathbf{Y})\partial_t|_{\boldsymbol{\chi}}\mathbf{Y} + \mathbf{A}_i(\mathbf{Y})\partial_{x_i}\mathbf{Y} + \mathbf{C}(\mathbf{Y})\mathbf{Y} = \mathbf{0} \tag{B2}$$

with

$$\mathbf{A}_0^{(\mathrm{NG})} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ v_1 & \rho & 0 & 0 & 0 \\ v_2 & 0 & \rho & 0 & 0 \\ v_3 & 0 & 0 & \rho & 0 \\ E & \rho v_1 & \rho v_2 & \rho v_3 & \rho e_{,p} \end{bmatrix} \tag{B3}$$

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$$\mathbf{C}^{(\mathrm{NG})} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -g_1 & 0 & 0 & 0 & 0 \\ -g_2 & 0 & 0 & 0 & 0 \\ -g_3 & 0 & 0 & 0 & 0 \\ -s & -\rho g_1 & -\rho g_2 & -\rho g_3 & 0 \end{bmatrix} \tag{B4}$$

and, for $i = 1, 2, 3$,

$$\mathbf{A}_i^{(\mathrm{NG})} = \left[\begin{array}{c|c|c|c|c} c_i & \rho\delta_{1i} & \rho\delta_{2i} & \rho\delta_{3i} & 0 \\ \hline c_i v_1 & \rho c_i + \rho v_1 \delta_{1i} & \rho v_1 \delta_{2i} & \rho v_1 \delta_{3i} & \delta_{1i} \\ \hline c_i v_2 & \rho v_2 \delta_{1i} & \rho c_i + \rho v_2 \delta_{2i} & \rho v_2 \delta_{3i} & \delta_{2i} \\ \hline c_i v_3 & \rho v_3 \delta_{1i} & \rho v_3 \delta_{2i} & \rho c_i + \rho v_3 \delta_{3i} & \delta_{3i} \\ \hline c_i E & \rho c_i v_1 + & \rho c_i v_2 + & \rho c_i v_3 + & \rho c_i e_{,p} + v_i \\ & (\rho E + p - \rho^2 e_{,\rho})\delta_{1i} & (\rho E + p - \rho^2 e_{,\rho})\delta_{2i} & (\rho E + p - \rho^2 e_{,\rho})\delta_{3i} & \end{array}\right] \tag{B5}$$

B.1.2. Galilean invariant approach. The previous approach is not the only way to derive a quasi-linear form of the Euler equations. Casting the Euler equations in advective form, the following set of invariant flux Jacobians is readily obtained:

$$\mathbf{A}_0^{(\mathrm{Gal})} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \rho & 0 & 0 & 0 \\ 0 & 0 & \rho & 0 & 0 \\ 0 & 0 & 0 & \rho & 0 \\ 0 & 0 & 0 & 0 & \rho e_{,p} \end{bmatrix}, \quad \mathbf{C}^{(\mathrm{Gal})} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -g_1 & 0 & 0 & 0 & 0 \\ -g_2 & 0 & 0 & 0 & 0 \\ -g_3 & 0 & 0 & 0 & 0 \\ -s & 0 & 0 & 0 & 0 \end{bmatrix} \tag{B6}$$

and, for $i = 1, 2, 3$,

$$\mathbf{A}_i^{(\mathrm{Gal})} = \left[\begin{array}{c|c|c|c|c} c_i & \rho\ \delta_{1i} & \rho\ \delta_{2i} & \rho\ \delta_{3i} & 0 \\ \hline 0 & \rho c_i & 0 & 0 & \delta_{1i} \\ \hline 0 & 0 & \rho c_i & 0 & \delta_{2i} \\ \hline 0 & 0 & 0 & \rho c_i & \delta_{3i} \\ \hline 0 & (p - \rho^2 e_{,\rho})\delta_{1i} & (p - \rho^2 e_{,\rho})\delta_{2i} & (p - \rho^2 e_{,\rho})\delta_{3i} & \rho c_i e_{,p} \end{array}\right] \tag{B7}$$

B.2. Density-internal energy variables

It is of interest, especially for the community developing shock hydrodynamics algorithms (*hydrocodes*, in short), the set of variables $\mathbf{Y} = [\rho\ \mathbf{v}^{\mathrm{T}}\ e]^{\mathrm{T}}$.

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B.2.1. The 'standard', non-invariant approach.

$$\mathbf{A}_0^{(\mathrm{NG})} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ v_1 & \rho & 0 & 0 & 0 \\ v_2 & 0 & \rho & 0 & 0 \\ v_3 & 0 & 0 & \rho & 0 \\ E & \rho v_1 & \rho v_2 & \rho v_3 & \rho \end{bmatrix} \tag{B8}$$

$$\mathbf{C}^{(\mathrm{NG})} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -g_1 & 0 & 0 & 0 & 0 \\ -g_2 & 0 & 0 & 0 & 0 \\ -g_3 & 0 & 0 & 0 & 0 \\ -s & -\rho g_1 & -\rho g_2 & -\rho g_3 & 0 \end{bmatrix} \tag{B9}$$

and, for $i = 1, 2, 3$,

$$\mathbf{A}_i^{(\mathrm{NG})} = \left[\begin{array}{c|c|c|c|c} c_i & \rho\ \delta_{1i} & \rho\ \delta_{2i} & \rho\ \delta_{3i} & 0 \\ \hline c_i v_1 + p_{,\rho}\delta_{1i} & \rho c_i + \rho v_1 \delta_{1i} & \rho v_1 \delta_{2i} & \rho v_1 \delta_{3i} & p_{,e}\delta_{1i} \\ \hline c_i v_2 + p_{,\rho}\delta_{2i} & \rho v_2 \delta_{1i} & \rho c_i + \rho v_2 \delta_{2i} & \rho v_2 \delta_{3i} & p_{,e}\delta_{2i} \\ \hline c_i v_3 + p_{,\rho}\delta_{3i} & \rho v_3 \delta_{1i} & \rho v_3 \delta_{2i} & \rho c_i + \rho v_3 \delta_{3i} & p_{,e}\delta_{3i} \\ \hline c_i E + p_{,\rho} v_i & \rho c_i v_1 + & \rho c_i v_2 + & \rho c_i v_3 + & \rho c_i + p_{,e} v_i \\ & (\rho E + p)\delta_{1i} & (\rho E + p)\delta_{2i} & (\rho E + p)\delta_{3i} & \end{array}\right] \tag{B10}$$

B.2.2. Galilean invariant approach.

$$\mathbf{A}_0^{(\mathrm{Gal})} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \rho & 0 & 0 & 0 \\ 0 & 0 & \rho & 0 & 0 \\ 0 & 0 & 0 & \rho & 0 \\ 0 & 0 & 0 & 0 & \rho \end{bmatrix}, \quad \mathbf{C}^{(\mathrm{Gal})} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -g_1 & 0 & 0 & 0 & 0 \\ -g_2 & 0 & 0 & 0 & 0 \\ -g_3 & 0 & 0 & 0 & 0 \\ -s & 0 & 0 & 0 & 0 \end{bmatrix} \tag{B11}$$

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and, for $i = 1, 2, 3$,

$$\mathbf{A}_i^{(\mathrm{Gal})} = \left[\begin{array}{c|c|c|c|c} c_i & \rho\ \delta_{1i} & \rho\ \delta_{2i} & \rho\ \delta_{3i} & 0 \\ \hline p_{,\rho}\ \delta_{1i} & \rho c_i & 0 & 0 & p_{,e}\ \delta_{1i} \\ \hline p_{,\rho}\ \delta_{2i} & 0 & \rho c_i & 0 & p_{,e}\ \delta_{2i} \\ \hline p_{,\rho}\ \delta_{3i} & 0 & 0 & \rho c_i & p_{,e}\ \delta_{3i} \\ \hline 0 & p\ \delta_{1i} & p\ \delta_{2i} & p\ \delta_{3i} & \rho c_i \end{array}\right] \tag{B12}$$

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